Research Statement

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My main research so far has been in the theory of Markov numbers, integer geometry, and knot theory. I am currently working on problems in geometric continued fractions. Below I give a short overview of my research so far, along with some open problems in the areas. Along with my current research I am looking forward to studying new and interesting problems in number theory and related areas.

1 Markov numbers

I give a small background to Markov numbers in Subsection 1.1. My main results are contained in Subsections 1.2 and 1.3, with an example in Subsection 1.4.

1.1 Background

Let us discuss Markov numbers and reduced forms.

1.1.1 What are Markov numbers?

Regular Markov numbers are the positive integers in the solutions to the Markov Diophantine equation

$$x^2 + y^2 + z^2 = 3xyz.$$

The study of regular Markov numbers has a rich history, beginning with link between these numbers, minima of binary quadratic forms, and the *worst approximable numbers*, given by A. Markov in the late 19^{th} century.

1.1.2 Reduced forms

Let $f(x, y) = ax^2 + bxy + cy^2$ be a binary quadratic form with positive discriminant $b^2 - 4ac > 0$. We call the minimal value that a form attains at integer points a *general Markov number*. We call f reduced if it splits into factors

$$f(x, y) = \lambda(y - r_1x)(y - r_2x)$$

for some real numbers $\lambda \neq 0$, $r_1 > 1$, and $-1 < r_2 < 0$. Let α be a rational number and

$$\alpha = a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_n}}}$$

for some integer a_1 and positive integers $a_2, \ldots, a_n$. The right hand side of this expression is called the *continued fraction of* α and is written $[a_0; a_1 : \ldots : a_n]$. Let

$$[a_1; a_2 : \ldots : a_{2n}] = \frac{p_1}{q_1},$$
$$[a_1; a_2 : \ldots : a_{2n-1}] = \frac{p_2}{q_2}.$$

We call

$$f_\alpha(x, y) = p_2 x^2 + (p_1 - q_2)xy - q_2 y^2$$

the *reduced form for* α.

1.2 Computation of Markov numbers

Any form is $\mathrm{SL}(2, \mathbb{Z})$ equivalent to a reduced form. This is an equivalence relation. For $\alpha = [a_1; \ldots : a_n]$ there are at most n reduced forms in the equivalence class of f_α. At least one of these reduced forms will attain it's minimal value at integer points at the point $(1, 0)$. We have generated trees of continued fractions α, whose forms f_α attain their minimum (their general Markov number) at $(1, 0)$.

Let $\alpha = (a_1, \ldots, a_n)$, $\beta = (b_1, \ldots, b_m)$, and $\alpha\beta = (a_1, \ldots, a_n, b_1, \ldots, b_m)$. We have proved in the joint paper with O. Karpenkov [4] that *under certain conditions for sequences* α *and* β *where the forms* f_α *and* f_β *are minimal at* $(1, 0)$*, we have that the form* $f_{\alpha\beta}$ *is also minimal at* $(1, 0)$.

Sequences satisfying the conditions for this statement are called *Markov sequences.* These are finite sequences of positive integers. These sequences are also known as Christoffel words or Cohn words.

I have proven one such condition needed for a sequence to be a Markov sequence in [1], that *there exists a cyclic shift of each Markov sequence that is palindromic.* Moreover, I have constructed the cyclic shift for required each sequence.

Continued fractions of Markov sequences are related to values on the Markov spectrum by an identity of O. Perron. The value is called the *Markov value of the sequence.* I would like to explore this connection more.

1.3 Uniqueness conjecture for general Markov number numbers

Recall that the *uniqueness conjecture for regular Markov numbers*, first stated by G. Frobenius in 1913, says the following; given a Markov triple (a, M, b) where a, $b \leq M$, then a and b are defined uniquely by M.

I have studied a similar statement for triples of general Markov numbers in my PhD thesis [2]. Using the recurrence relations defined in [4], we generated triples of general Markov numbers. I have shown (in the paper [2]) that there can be general Markov triples that are not uniquely determined by their largest elements. Hence *the uniqueness conjecture for general Markov numbers is false.* To do this I checked numerous examples computationally. This tells us for instance that the structure of Markov sequences has no bearing on whether the uniqueness conjecture for regular Markov numbers is true or not. There remain many open questions to study in this area.

1.4 Counter example to the general uniqueness conjecture

Consider the sequences built by concatenation.

$$\begin{array}{ccccccccc}
\alpha & & & & & & & & \beta \\
\alpha & & & & \alpha\beta & & & & \beta \\
\alpha & & \alpha\alpha\beta & & \alpha\beta & & \alpha\beta\beta & & \beta \\
\alpha & \alpha\alpha\alpha\beta & \alpha\alpha\beta & \alpha\alpha\beta\alpha\beta & \alpha\beta & \alpha\beta\alpha\beta\beta & \alpha\beta\beta & \alpha\beta\beta\beta & \beta
\end{array}$$

Continuing this concatenation indefinitely for any pair of palindromic sequences α and β creates a family of sequences $\{\gamma_i \mid i \geq 0\}$ with forms f_{γ_i}. These families of sequences allow quick computation of general Markov numbers, since *the minima of the forms* f_{γ_i} *are attained at* $(1,0)$.

If $\alpha = (1,1)$ and $\beta = (2,2)$ then these are exactly the sequences related to regular Markov numbers (for any one of these sequences γ the value $f_\gamma(1,0)$ is a regular Markov number). The general uniqueness conjectures say for such a family of sequences $\{\gamma_i \mid i \geq 0\}$ that the values $f_{\gamma_i}(1,0)$ are different for each $i \geq 0$.

Let $\alpha = (4,4)$ and $\beta = (11,11)$. Consider the two sequences

$$\begin{aligned}
\gamma_1 &= \alpha\beta\beta\beta\beta = (4,4,11,11,11,11,11,11,11,11), \\
\gamma_2 &= \alpha\alpha\alpha\alpha\alpha\alpha\beta = (4,4,4,4,4,4,4,4,4,4,4,4,11,11).
\end{aligned}$$

Then the forms f_{γ_1} and f_{γ_2} both attain their minimum at the point $(1,0)$, and the values are

$$f_{\gamma_1}(1,0) = f_{\gamma_2}(1,0) = 355318099.$$

This is *a counter example to the general uniqueness conjecture for* $(4,4)$ *and* $(11,11)$.

2 Integer geometry

In the joint paper with my supervisor O. Karpenkov [3] we have studied the relation between integer geometry and continued fractions. The endpoints of a finite broken line are joined to the origin by two rays. These rays define a binary quadratic form f. Integer geometry allows one to assign an $\mathrm{SL}(2,\mathbb{Z})$ invariant sequence of numbers, called an *LLS sequence*, to a broken line, see Figure 1. We show in [3] that sums of certain continued fractions of this LLS sequence give the value of the form f at the vertices of the broken line.

For instance, if the broken line with vertices $A_0 \dots A_{n+m}$ has LLS sequence $(a_0, \dots, a_{2n+2m})$ then

$$f(A_n) = \left| \frac{\sqrt{\Delta(f)}}{a_{2n} + [0; a_{2n-1} : a_{2n-2} : \dots : a_0] + [0; a_{2n+1} : a_{2n+2} : \dots : a_{2n+2m}]} \right|.$$

Recently with O. Karpenkov I have started to study an extended Euclidean algorithm based on Voronoi continued fractions. The problem of finding a natural stopping point for the algorithm (if one even exists) is still open.

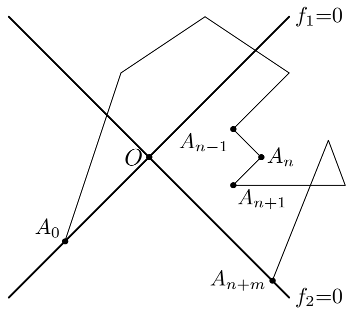

Figure 1

3 Knot theory

During my master's thesis [1] I studied a problem relating to the knot theory. Based on the work of J. O'Hara, the *Möbius energy* of a knot is a property that can help determine when two loops in space are isotopic. We attempted to create a computer algorithm to minimize the Möbius energy of obstacle avoiding loops.

Publications

[1] M. Van Son. Palindromic Sequences of the Markov Spectrum. *Mathematical Notes*, 106:457–467, 2019.

[2] M. Van Son. Uniqueness conjectures for extended Markov numbers. *arXiv:1911.00746, preprint*, 2019.

[3] O. Karpenkov and M. Van Son. Generalized Perron Identity for broken lines. *Journal de théorie des nombres de Bordeaux*, 31:131–144, 2019.

[4] O. Karpenkov and M. Van Son. Generalised Markov numbers. *Journal of Number Theory*, 213:16–66 2020.